European Finance Review **1:** 81–104, 1997.

Two Closed-Form Formulas for the Futures Price in the Presence of a Quality Option

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Abstract. The paper derives closed-form formulas for the futures price in the presence of a multi-asset quality option. This is done for two cases: In the first one the underlying assets are zero coupon bonds with different maturities in the single-factor Vasicek model. In the second one these are commodities in a multi-factor setting, again with Vasicek interest rate uncertainty.

Key words: Futures contracts, quality option, Vasicek model

1. Introduction

The US Treasury Bond futures contract traded on the Chicago Board of Trade contains four options, which are possessed by the short. These are the quality option, the accrued interest option, the wild card option and the end-of-the-month option. A comprehensive survey by Chance and Hemler (1993) reviews the literature on the impact of these options on bond futures prices.

Because of the complicated nature of the contract, it is a common practice in this literature to focus on 'one option at a time', that is, to analyze a bond futures contract which possessed only one of these options. (A recent example is Cohen (1995).) In spite of the fact that this may be too simplistic relative to the real world, it does provide important qualitative insights and it can even be applicable in situations when one option dominates in importance.

This is also what is done in this paper, where a futures contract endowed with the quality option will be studied. To recapitulate, this option allows the short trader to choose which asset to deliver from a list of eligible assets, with a certain price adjustment. In the CBOT Treasury bond futures contract, these assets are Treasury bonds with at least 15 years to maturity or first call date. However, our purpose in this paper is not to model this particular contract, but rather to better understand the quality option in general. A good way to achieve that, we believe, is via simple closed-form formulas. Such formulas are developed for two cases, both with Vasicek interest rate uncertainty: In the first one the underlying assets are zero coupon bonds with different maturities, while in the second one these are commodities.

While the main goal of the paper was to analyze the quality option, it evolved to more than that. The paper derives a new compact futures PDE which is applicable when the futures boundary condition is a function of other futures prices. The quality option is actually only a special case. From a mechanical point of view, all the work in this paper is PDE-based, where the point of departure is the fundamental futures PDE, which is taken as known. The assumed stochastic processes of the state variables are needed only to determine the coefficients of the PDEs, but otherwise stochastic calculus is not used. Thus one pedagogical contribution of the paper is that it demonstrates how to use 'classical' PDE methods as an alternative to the equivalent martingale measure technique which now dominates the literature. We believe that, once the reader is past the preparatory (and general) material in Section 3 below, then the actual work of solving the appropriate PDE for a given problem via standard methods may sometimes be easier than computing risk-neutralized expectations. (See for example Section 5 below. Admittedly, ease is a subjective matter.)

The paper is similar in spirit to an earlier working paper by Carr and Chen (1993), who study bond futures contracts (endowed with the quality option) in a one- and two-factor interest rate models like in Cox, Ingersoll and Ross (1985, henceforth CIR), and also empirically test these models. Their formulas are expressed in terms of the futures prices of the underlying assets. This is also what is done here, except that our setting is simpler and our methods are different.

Another related paper, which provides a good analysis of the issues concerning the quality option, is by Ritchken and Sankarasubramanian (1992). What distinguishes their paper from the previous literature (see Boyle (1989) and references therein) is the explicit treatment of the T-bond contract as a (marked-to-market) futures contract, instead of viewing it as a (single payoff) forward contract. Among other things, they obtain, in a specific one-factor model (exponentially dampened volatility for the forward rate), a closed-form formula for the bond futures price when a quality option is present. However, in their formula the futures price is expressed in terms of the underlying bond prices, whereas we are interested, like Carr and Chen, in the connection to the underlying 'more primitive' futures prices. Such a connection, we believe, better demonstrates the effect of adding the quality option to a 'straight' futures contract.

Finally, a more recent publication on the quality option utilizing the CIR model is Cherubini and Esposito (1995). Lin and Paxson (1995) and Ritchken and Sankarasubramanian (1995) use the Heath-Jarrow-Morton (1992) setting.

The paper is organized as follows: The basic definitions of futures contracts and the quality option are reviewed in Section 2. The continuous-time economic setting and a new (transformed) futures PDE are introduced in Section 3. This is followed by Sections 4 and 5, which contain the main results – two closed-form formulas for the futures price in the presence of the quality option. The insights from these formulas are discussed in Section 6. Section 7 is a short summary.

2. Futures Prices and the Quality Option

Before introducing more specific assumptions on the economic environment, we recall that a 'straight' **futures contract** (not containing any options, and with instantaneous delivery at expiration) is a traded security whereby a given asset is delivered at a specified future time T, henceforth '**the delivery time**' or '**expiration**'. In addition, the futures price corresponding to the contract is determined by the market such that

(a) at any time, assuming a long or a short position requires zero cash outlay. That is, the value of the contract is always zero. The same is true for offsetting an existing position, except that

(b) The contract is continuously marked to market. That is, increases in the futures price are continuously credited to the (margin account of) the holder of the long position and debited to the holder of the short position (with the obvious sign convention for negative increases).[1] In what follows the units of account in which this is done will be referred to as 'dollars'.

(c) The futures price at time T is equal to the value at that time of the deliverable asset.

Property (c) is in fact an obvious no-arbitrage result, not a part of the definition, but it can be used to define a 'cash-settlement' futures contract, without necessarily having physical delivery: For any random variable $\tilde{H}_T$ whose realization is known at time T (i.e., which is measurable with respect to the sigma-algebra representing the information known at time T), a futures contract expiring at time T can be *defined* by the requirement that its futures price at that time will be equal to $\tilde{H}_T$. It means that marking to market is executed as follows: At expiration, $\tilde{H}_T$ is regarded as the last futures price for the purpose of computing the last cash settlement payoff, and at any other time $t < T$ this is done as usual with respect to a market-determined futures price.

In this paper we are interested in futures contracts where the short party has the **quality option** with a multiplicative price adjustment: A set of n 'eligible' assets can be delivered at the delivery time T. For each asset i from this set, a positive conversion factor of $1/k_i$ is specified when the contract starts trading ('time 0' below), so that by definition the 'invoice amount' paid upon its delivery is the contract's futures price Q prevailing in the market at time T times the conversion factor, i.e., Q/k_i dollars.[2] The short trader has the option to choose at time T which one of the n eligible assets to deliver. (The term 'quality option' comes from agricultural futures, where different grades of a certain grain can be delivered.) Such a contract may be called '**a futures contract endowed with a quality option**' or, more compactly, '**a quality futures contract**'.

Let $\tilde{P}_T^i$ be the spot price at the delivery time T of asset $i \in \{1, \ldots, n\}$, and let $\tilde{Q}_T$ be the futures price at that time of the quality futures contract. If there are no transaction costs, then in the absence of arbitrage opportunities this futures price must be the minimum of the ‘adjusted asset prices’, namely

$$\tilde{Q}_T = \min(k_1\tilde{P}_T^1, \ldots, k_n\tilde{P}_T^n). \tag{2.1}$$

If this is violated at time T, then there is ‘on-the-spot arbitrage’. This simple fact is recorded in Gay and Manaster (1984), Kane and Marcus (1986) and Duffie (1989, Section 9.4), among others. Thus, we obtain:

Key observation: The quality futures contract may be regarded as a plain cash-settlement contract, whose futures price at time T is *defined* by (2.1).

3. Futures PDEs

The above definition of a futures contract and the setting below are similar to the ones in Duffie and Stanton (1992), henceforth D–S. The economic setting is a frictionless financial market where a specified set of securities is traded continuously in a given finite or infinite time interval I starting from the origin. (All times mentioned below are assumed to be within this interval.) Uncertainty is represented by a probability space (Ω, $\mathcal{F}$, $\mathcal{P}$), where $\mathcal{P}$ is interpreted as the objective probability measure, and by a state vector $X_t \in \mathbb{R}^k$ satisfying the stochastic differential equation

$$\mathrm{d}X_t = \nu(X_t, t)\,\mathrm{d}t + \eta(X_t, t)\,\mathrm{d}B_t. \tag{3.1}$$

Here B is a k-dimensional standard Brownian motion w.r.t. $\mathcal{P}$ and $\nu : \mathbb{R}^k \times I \to \mathbb{R}^k$ and $\eta : \mathbb{R}^k \times I \to \mathbb{R}^{k\times k}$ are suitably well-behaved functions. D–S restrict themselves to prices and dividends of the form $f(X_t, t)$ for some suitably smooth f, and assume (informally stated) that there exist k securities which, together with the instantaneous riskless rate $r_t = R(X_t, t)$, span the k state variables. Such a Markovian structure is essential for the development of the futures PDE below. We rely on the D–S setting in order to avoid deriving the PDE from scratch. However, the reader who regards the PDE as known, in any other similar framework, will not need to consult D–S. For example, a futures PDE, derived by classical ‘infinitesimal no-arbitrage’ arguments, is available in Brennan and Schwartz (1985).

Our point of departure will be the fundamental futures PDE, and the results which follow are then obtained solely by PDE methods. We will also comment on the probabilistic interpretation, but, once the PDE is given, this is not necessary for the formal development. Let us start with the mathematical notation: For a suitably-smooth $f : \mathbb{R}^k \times I \to \mathbb{R}$, $f_x(x,t)$ is the row vector of partial derivatives $(f_{x_1}, \ldots, f_{x_k})$ with respect to the k state variables, $f_{xx}(x,t)$ is the matrix $(f_{x_i x_j})$ of second derivatives and, as usual, $f_t = \partial f(x,t)/\partial t$. (A t-subscript may also denote a time variable, as in (3.1). This will be clear from the context.) For $f : \mathbb{R}^k \times I \to \mathbb{R}^n$, $f_x(x,t)$ is the matrix whose ith row is $(f^i)_x$. The transpose of a matrix is denoted by $(\cdot)'$ and f'_x will mean $(f_x)'$. The trace of a matrix is represented by tr$[\cdot]$, and recall that tr$[AB] = \sum_i \sum_k A_{i,k} B_{k,i}$, where here A and B are any $m \times n$ and $n \times m$ matrices, respectively.

The known fact which we use is this: If the terminal futures price at time T of a futures contract is of the form $\bar{H}(X_T)$, then the futures price at time t is $H(X_t, t)$ dollars, where $H(x,t)$ is the solution of

$$H_t(x,t) + H_x(x,t)\mu(x,t) + \frac{1}{2}\,\mathrm{tr}[\chi H_{xx}] = 0, \tag{3.2}$$

$$H(x,T) = \bar{H}(x), \tag{3.3}$$

and where $\chi \equiv \eta\eta'$ and $\mu : \mathbb{R}^k \times I \to \mathbb{R}^k$ is some (column) vector function which is the same for all futures contracts. (For an exact expression for μ, and on the derivation of the PDE, see Appendix C.) Note that the last term of the PDE may also be written as $\frac{1}{2}\,\mathrm{tr}[\eta' H_{xx}\eta]$, which is the way it is written in D–S.[3]

For the quality futures contract described in the previous section, we will later solve the above PDE with the boundary condition

$$\begin{aligned} \bar{H}(x) &= \min(k_1 J^1(x,T;T), \ldots, k_n J^n(x,T;T)) \\ &= J^n(x,T;T) \cdot \min\left(k_1 \frac{J^1(x,T;T)}{J^n(x,T;T)}, \ldots, k_{n-1}\frac{J^{n-1}(x,T;T)}{J^n(x,T;T)}, k_n\right), \end{aligned} \tag{3.4}$$

where the second equality is valid if $J^n > 0$. Here $J^i(x,t;T)$ represents the futures price of asset i at time $t \leq T$, given that $X_t = x$, and we used the fact that at the delivery time T it is equal to the spot price.

REMARK 1. In Section 5 below it will be assumed that the Brownian motion vector satisfies (with informal notation) $\mathrm{d}B_t \cdot (\mathrm{d}B_t)' = \rho\,\mathrm{d}t$, where the $k \times k$ matrix ρ is a deterministic function of time, instead of assuming as above that ρ is the identity matrix. In this case futures prices also satisfy the PDE (3.2), except that here $\chi = \eta\rho\eta'$. This can be shown by modifying Itô's lemma in the original proof.

Next, we present a transformed futures PDE so that the new state variables are some fixed relative futures prices. Since it does not require more effort, we will write it in a form which is applicable to more general boundary conditions:

PROPOSITION [3-1]. Suppose that $J^1(x,t), \ldots, J^n(x,t)$ are solutions of the futures PDE (3.2). We assume that $n \geq 2$ and $J^n > 0$. Define $y : \mathbb{R}^k \times [0,T) \to \mathbb{R}^{n-1}$ by

$$y^i(x,t) \equiv J^i(x,t)/J^n(x,t), \quad i = 1, \ldots, n-1, \tag{3.5}$$

and assume that it satisfies the following property: There exists a matrix function $c : \mathbb{R}^{n-1} \times [0,T) \to \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$y_x(x,\, t)\chi(x,\, t)y_x'(x,\, t) = c(y(x,\, t),\, t), \tag{3.6}$$

where χ is as in (3.2). (See Remark 2 below.)

Now suppose $h : \mathbb{R}^{n-1} \times [0,T] \to \mathbb{R}$ is suitably smooth (i.e., an element of $C^{2,1}$; see Appendix A). Then, the futures PDE with a given boundary condition $\bar{H}$, as in (3.2)–(3.3), is solved by

$$H(x,t) = J^n(x,t)h(y(x,t),t) = J^n(x,t)h\left(\frac{J^1(x,t)}{J^n(x,t)},\ldots,\frac{J^{n-1}(x,t)}{J^n(x,t)},t\right), \tag{3.7}$$

if only if the following two conditions are satisfied:

(i) There exists $\bar{h} : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\bar{H}(x) = J^n(x,T)\bar{h}(y(x,T)). \tag{3.8}$$

(That is, $\bar{H}$ is of the form $\bar{H}(x) = f(J^1(x,T),\ldots,J^n(x,T))$, where f is homogeneous of degree 1.)

(ii) h solves for $y \in \mathbb{R}^{n-1}$, $t \in [0,T)$,

$$h_t(y,t) + \frac{1}{2}\,\mathrm{tr}[c(y,t)h_{yy}(y,t)] = 0, \tag{3.9}$$

i.e.,

$$\frac{\partial h}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n-1} c_{i,j}(y,t)\frac{\partial^2 h}{\partial y_i \partial y_j} = 0, \tag{3.9$'$}$$

with

$$h(y,T) = \bar{h}(y). \tag{3.10}$$

Proof. Apply Lemma [A-2] from Appendix A. □

REMARK 2. The purpose of the condition associated with (3.6) is to ensure that the original PDE (3.2) can be transformed into a well-defined PDE in the new variables $y^1,\ldots,y^{n-1}$. This is an invertibility condition which may or may not be satisfied. The probabilistic interpretation, which we do not use explicitly, is this: Informally stated, the $(n-1)$-dimensional stochastic process $Y_t \equiv y(X_t,t)$ satisfies

$$\mathrm{d}Y_t \cdot \mathrm{d}Y_t' = y_x(X_t,t)\chi(X_t,t)y_x'(X_t,t)\,\mathrm{d}t. \tag{3.11}$$

The above condition says that this variance-covariance matrix can be written as a function of (Y_t,t). The conclusion (3.9) can then be obtained by constructing a new equivalent martingale measure $\mathcal{Q}^n$ (along the lines of Geman, El Karoui and Rochet (1995) and Duffie (1996)) under which normalized futures prices, with $J^n(X_t, t)$ as the numeraire, are martingales. Equation (3.9) is the expression of the fact that the drift of a futures price in this case must be zero.

REMARK 3. Paradoxically, the proposition also covers the case where f (from (i)) is not necessarily homogeneous of degree 1. This is because it can be applied to $J^1(x,t), \ldots, J^n(x,t),\ J^{n+1} \equiv 1$, with $y^i = J^i/J^{n+1}$. This will be demonstrated in the proof of Proposition [6-2].

REMARK 4. In Section 5 we will need the following observation, which is just a restatement of Equation (3.7) when now H and h are expressed as integrals of $\bar{H}$ and $\bar{h}$, respectively, against the appropriate fundamental solutions (Green functions): Suppose $G(x,t;x',T)$ and $g(y,t;y',T)$ are the fundamental solutions of the PDEs (3.2) and (3.9), respectively (Here the prime denotes another variable, not a transpose.). Suppose $\bar{H}$ and $\bar{h}$ are as in (3.8), where $\bar{h}$ is continuous and suitably well-behaved. (See Duffie (1996), Appendix E and references therein.) Then

$$\int_{\mathbb{R}^n} G(x,t;x',T)\bar{H}(x')\,\mathrm{d}x' = J^n(x,t)\int_{\mathbb{R}^{n-1}} g(y(x,t),t;y',T)\bar{h}(y')\,\mathrm{d}y'. \tag{3.12}$$

4. The Quality Futures Contract on n Zero Coupon Bonds in the Vasicek Setting

In Vasicek (1977), it is assumed that the instantaneous riskless rate $\{r_t; t \geq 0\}$ is the only state variable for the bond prices and their derivative securities. It is assumed that it evolves according to

$$\mathrm{d}r_t = \alpha(r_0 - r)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \tag{4.1}$$

where $\alpha > 0$, r_0 and $\sigma > 0$ are constants, and $\{B_t\}$ is a one-dimensional Brownian motion relative to a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. (Again, $\mathcal{P}$ is interpreted as the objective probability measure.) As in Section 5 in Vasicek's paper, we will assume further that "the price of risk" λ is a constant. For our purposes it suffices to reformulate it as the following assumption: μ from the futures PDE (3.2) (and Appendix C) is given by $\mu = \alpha(r_0 - r) - \sigma\lambda$ where λ is a constant. Like Jamshidian (1989) we will prefer the notation $\mu = \alpha(\bar{r} - r)$, where $\bar{r} \equiv r_0 - \sigma\lambda/\alpha$. This means that the futures PDE is

$$H_t + \alpha(\bar{r} - r)H_r + \frac{1}{2}\sigma^2 H_{rr} = 0. \tag{4.2}$$

Let us start with the T-expiration **futures-on-the-spot-rate contract**. This contract is defined via the boundary condition $\bar{H} = r$. That is, this is a cash-settlement contract whose futures price at time T is defined to be r_T. Its futures price at time t, given that $r_t = r$, will be denoted $M(r, t; T)$.

PROPOSITION [4-1].

$$M(r, t, T) = e^{-\alpha(T-t)}r + (1 - e^{-\alpha(T-t)})\bar{r}. \tag{4.3}$$

Proof. It is straightforward to verify that (4.3) is a solution of (4.2) with $\bar{H}(r) = r$, and the proof is finished.

Parenthetically, (4.3) was constructed by trying a solution of the form $\varphi(t)r + \psi(t)$, such that $\varphi(T) = 1$, $\psi(T) = 0$. Substitution in (4.2) gives, after rearrangement,

$$r\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t} - \alpha\varphi\right) + \frac{\mathrm{d}\psi}{\mathrm{d}t} + \alpha\bar{r}\varphi = 0. \tag{4.4}$$

Now one solves $\mathrm{d}\varphi/\mathrm{d}t - \alpha\varphi = 0$ and then $\mathrm{d}\psi/\mathrm{d}t + \alpha\bar{r}\varphi = 0$, with the appropriate boundary conditions. □

Note that the diffusion coefficient in (4.1) does not play any role in the proof, which is valid therefore in other settings, for example CIR's. We believe that our techniques below may also be applied in more general cases, namely for the larger family of affine single factor term structure models (see Duffie (1996, Section 7E)) which include the Vasicek and the CIR models. This will not be attempted here.

Next, let us turn to the futures contract on a zero coupon bond. For any 'fixed' $0 < T < s$, we can define **the** (T, s)**-bond futures contract**, whereby the s-maturity unit zero coupon bond is delivered at time T. Let $P(r, t; s)$ (for $t \leq s$) and $F(r, t; T, s)$ (for $t \leq T$) be, respectively, the corresponding spot price and the futures price at time t, given that $r_t = r$. It is known (see Vasicek's paper) that P is the solution of

$$P_t + \alpha(\bar{r} - r)P_r + \frac{1}{2}\sigma^2 P_{rr} - rP = 0 \tag{4.5}$$

with the boundary condition $P(r, s; s) = 1$, and it is given by[4]

$$\begin{aligned} P(r, t; s) &= A_0(t, s)\exp(\bar{r}(b(t, s) - (s - t)))\exp(-rb(t, s)) \\ &= A(t, s)\exp(-rb(t, s)) \end{aligned} \tag{4.6}$$

where

$$b(t, s) \equiv \frac{1}{\alpha}(1 - e^{-\alpha(s-t)}), \tag{4.7}$$

$$A_0(t,s) \equiv \exp\left(\frac{\sigma^2}{2\alpha^2}(s-t-b(t,s)) - \frac{\sigma^2}{4\alpha}b^2(t,s)\right), \tag{4.8}$$

and where $A(t,s)$ is *defined* via the second equality in (4.6). (Both representations in (4.6) will be useful below.)

Next, let us find F:

PROPOSITION [4-2]. Let $0 < T \leq s$. The (T,s)-bond futures price is

$$F(r,t;T,s) = A(T,s)\cdot \exp(-b(T,s)M(r,t,T) + \frac{1}{2}b^2(T,s)v^2(t,T)). \tag{4.9}$$

where b, A and M are functions as above and

$$v^2(t,T) \equiv \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha(T-t)}) = \sigma^2\int_t^T e^{-2\alpha(T-u)}\,\mathrm{d}u. \tag{4.10}$$

Proof. Let us apply Proposition [3-1], where now M becomes the new state variable. To achieve that, take $J^1 \equiv M$, $J^2 \equiv 1$, $y = J^1/J^2 = M$. Equation (4.3) gives $y_r = e^{-\alpha(T-t)}$, which means that c from (3.6) is now $\sigma^2 e^{-2\alpha(T-t)}$, and the transformed PDE (3.9′) becomes

$$h_t(y,t) + \frac{1}{2}\sigma^2 e^{-2\alpha(T-t)}h_{yy}(y,t) = 0. \tag{4.11}$$

Now fix $s \geq T$. In light of (4.6), the boundary condition at time T, namely $\bar{H}(r) = P(r,T;s) = P(M(r,T,T),T;s)$, translates to

$$h(y,T) = \bar{h}(y) \equiv A(T,s)e^{-yb(T,s)}. \tag{4.12}$$

The system (4.11)–(4.12) can be solved by trying a solution of the form $h = \bar{h}(y)\psi(t)$, with the above known $\bar{h}$ and with $\psi(T) = 1$. This easily yields (details are left to the reader)

$$\psi = \exp\left(\frac{1}{2}b^2(T,s)v^2(t,T)\right),$$

and the desired result follows. □

The following proposition gives a Black–Scholes-type expression for the futures price of the bond contract endowed with the quality option, with two eligible bonds.

PROPOSITION [4-3]. For $t < T \leq s_1 < s_2$, let $F(r,t;T,s_i)$ be the T-expiration futures price at time t, given that $r_t = r$, on the s_i-maturity unit zero coupon bond. Then the futures price of the quality futures contract on these two bonds is

$$Q(r,t;T,s_1,s_2) = k_1F(r,t;T,s_1)\cdot N(d_{1,2}) + k_2F(r,t;T,s_2)\cdot N(d_{2,1})$$

(4.13)

where $1/k_i$, $i = 1, 2$, are the conversion factors, $N(\cdot)$ is the cumulative distribution function of the standard normal variable,

$$d_{i,j} = d_{i,j}(r, t; T) = \frac{\ln \dfrac{k_j F(r, t; T, s_j)}{k_i F(r, t; T, s_i)} - \dfrac{1}{2}\Lambda^2(t; T, s_i, s_j)}{\Lambda(t; T, s_i, s_j)} \tag{4.14}$$

for $i \neq j$ in $\{1, 2\}$, and where $\Lambda(t; T, s_i, s_j)$ is the positive square root of

$$\Lambda^2(t; T, s_i, s_j) = [b(T, s_j) - b(T, s_i)]^2 v^2(t, T). \tag{4.15}$$

Here v^2 is as in (4.10), and we note that $b(T, s_2) > b(T, s_1)$ by (4.7).

Proof. Denote, for brevity, $A_i = A(T, s_i)$, $b_i = b(T, s_i)$, $i = 1, 2$. To apply Proposition [3-1], take

$$\begin{aligned} y(r, t) &\equiv \frac{F(r, t; T, s_1)}{F(r, t; T, s_2)} \\ &= \frac{A_1}{A_2} \cdot \exp\left((b_2 - b_1)M(r, t, T) - \frac{1}{2}[(b_2)^2 - (b_1)^2]v^2(t, T)\right) \end{aligned} \tag{4.16}$$

Thus $y_r = (b_2 - b_1)\exp(-\alpha(T - t)) \cdot y$, which means that c from (3.6) is given by

$$c(y, t) = \sigma^2(b_2 - b_1)^2 \exp(-2\alpha(T - t)) \cdot y^2, \tag{4.17}$$

and the PDE (3.9′) and the boundary condition (3.4) become, denoting $z^+ \equiv \max(z, 0)$,

$$h_t(y, t) + \frac{1}{2}\sigma^2(b_2 - b_1)^2 e^{-2\alpha(T-t)} y^2 h_{yy}(y, t) = 0, \tag{4.18}$$

$$h(y, T) = \bar{h}(y) \equiv \min(k_1 y, k_2) = k_1(y - (y - k_2/k_1)^+). \tag{4.19}$$

One way to solve (4.18)–(4.19) is as follows: First solve the PDE with the boundary condition $(y - k_2/k_1)^+$. The result is the well-known Black–Scholes formula for a call option under ‘zero interest rate and time-dependent deterministic volatility’. Then use the third equality in (4.19) to conclude the argument. The solution is

$$h(y, t) = k_1 y N(D_1(y, t)) + k_2 N(D_2(y, t)), \tag{4.20}$$

where

$$D_1(y, t) = -[\ln(k_1 y/k_2) + \tfrac{1}{2}\Lambda^2(t)]/\Lambda(t) = [\ln(k_2/k_1 y) - \tfrac{1}{2}\Lambda^2(t)]/\Lambda(t), \tag{4.21}$$

$$D_2(y,t) = -D_1(y,t) - \Lambda(t) = [\ln(k_1 y/k_2) - \tfrac{1}{2}\Lambda^2(t)]/\Lambda(t), \tag{4.22}$$

and $\Lambda(t) = \Lambda(t; T, s_1, s_2)$ is as in (4.15). Going back from h to H, as in (3.7), gives the desired result. □

REMARK 5: Propositions [4-1] and [4-2] can also be proved by risk-neutral valuation, namely by the equivalent martingale measure technique. This is demonstrated in an appendix which was omitted from this version of the paper, but is available from the author.

Consider now the more general case where the quality futures contract is on n zero-coupon bonds, with maturities $s_1 < s_2 < \cdots < s_n$. For $i = 1, \ldots, n$, let

$$\mathcal{M}_i = \{r \in \mathbb{R}; k_i P(r,T;s_i) < k_j P(r,T;s_j)\ \forall j \neq i\}.$$

As functions of r, adjusted bond prices $\{k_j P(r,T;s_j)\}$ intersect one another only once. Thus each $\mathcal{M}_i$ is an intersection of open intervals, which is an open interval or an empty set. Because there is a finite number of intersection points, one can identify by inspection the indices i for which $\mathcal{M}_i$ is empty. Removing such redundant elements from the given set of eligible-to-deliver bonds will clearly not affect the solution of our problem, hence we will assume in what follows that $\mathcal{M}_i$ is non-empty for each $i = 1, \ldots, n$. Thus $\{\mathcal{M}_i; i = 1, \ldots, n\}$ is a family of disjoint open intervals such that their closures exhaust $\mathbb{R}$. The idea of using such a partition appears, under different interest rate dynamics, in Ritchken and Sankarasubrahmanian (1992), Carr and Chen (1993) and Cherubini and Esposito (1995).

As before, we denote $A_i \equiv A(T, s_i)$, $b_i \equiv b(T, s_i)$, $i = 1, \ldots, n$. Let us also denote $\bar{P}_i(r) \equiv k_i P(r,T;s_i)$. The sets $\{\mathcal{M}_i\}$ can be characterized as follows:

LEMMA [4-4]. For $i \neq j$, let $r_{i,j}$ be the intersection point of $\bar{P}_i(\cdot)$ and $\bar{P}_j(\cdot)$, i.e.,

$$r_{i,j} \equiv [\ln(k_j A_j / k_i A_i)]/(b_j - b_i). \tag{4.23}$$

Then:

(i) $r_{1,2} < r_{2,3} < \cdots < r_{n-1,n}$.
(ii) $\mathcal{M}_1 = (-\infty, r_{1,2}) = \{r \in \mathbb{R};\ \bar{P}_1(r) < \bar{P}_2(r)\}$.
$\mathcal{M}_i = (r_{i-1,i},\ r_{i,i+1}) = \{r \in \mathbb{R};\ \bar{P}_i(r) < \bar{P}_j(r)$ for $j = i-1,\ i+1\}$ for $i = 2, \ldots, n-1$.
$\mathcal{M}_n = (r_{n-1,n}, \infty) = \{r \in \mathbb{R};\ \bar{P}_n(r) < \bar{P}_{n-1}(r)\}$.

Proof. It is convenient to denote, for $i \neq j$,

$$q_{i,j}(r) \equiv \bar{P}_j(r)/\bar{P}_i(r) = \exp(\ln(k_j A_j / k_i A_i) - r(b_j - b_i)),$$

so that $q_{i,j}(r_{i,j}) = 1$. Let us start with part (ii), with the equalities on the right. They can now be written as

$$(-\infty, r_{1,2}) = q_{1,2}^{-1}(1, \infty), \tag{4.24}$$

$$(r_{i-1,i},\infty)\cap(-\infty,r_{i,i+1}) = q_{i,i-1}^{-1}(1,\infty)\cap q_{i,i+1}^{-1}(1,\infty), \tag{4.25}$$

and

$$(r_{n-1,n},\infty) = q_{n,n-1}^{-1}(1,\infty). \tag{4.26}$$

We note that for $i<j$ we have $b_i<b_j$, and thus $q_{i,j}(\cdot)$ is decreasing and $q_{j,i}(\cdot)$ is increasing, and thus (4.26)–(4.28) are immediate.

Before completing the proof of (ii), let us prove (i). To that end, write

$$\mathcal{M}_i \subset q_{i,i-1}^{-1}(1,\infty)\cap q_{i,i+1}^{-1}(1,\infty) = (r_{i-1,i},r_{i,i+1}), \quad i=2,\ldots,n-1, \tag{4.27}$$

where the inclusion is clear from the definition of $\mathcal{M}_i$, and the equality is as in (4.25). Now the desired result follows from the assumption that each $\mathcal{M}_i$ is non-empty.

Lastly, we need to prove the equalities on the left in part (ii). Indeed, in analogy to (4.27) we have

$$\mathcal{M}_1 \subset (-\infty,r_{1,2}) \text{ and } \mathcal{M}_n \subset (r_{n-1,n},\infty). \tag{4.28}$$

Part (i) says that the open intervals $(-\infty,r_{1,2})$, $(r_{1,2},r_{2,3}),\ldots,(r_{n,n+1},\infty)$ are disjoint and the union of their closures is equal to $\mathbb{R}$. As it was pointed out, the same is true for $\{\mathcal{M}_i\}$. This means that the inclusions in (4.27) and (4.28) must actually be equalities. □

Note that our standing assumption that all $\{\mathcal{M}_i\}$ are non-empty is needed in the proof. The important conclusion is this: If an adjusted bond price $\bar{P}_i(\cdot)$ is smaller at $r\in\mathbb{R}$ than the adjusted bond prices with adjacent maturities, then it is smaller than all the other adjusted bond prices. In addition,

COROLLARY. Fix $r\in\mathbb{R}$. Suppose the minimum of $\{\bar{P}_1(r),\ldots,\bar{P}_n(r)\}$ is achieved by $\bar{P}_l(r)$. Then

$$\bar{P}_l(r)\le\bar{P}_{l+1}(r)\le\cdots\le\bar{P}_n(r),$$

$$\bar{P}_l(r)\le\bar{P}_{l-1}(r)\le\cdots\le\bar{P}_1(r).$$

As an immediate algebraic result, for each $r\in\mathbb{R}$,

$$\mathrm{Min}\{\bar{P}_1(r),\ldots,\bar{P}_n(r)\} = \sum_{j=1}^{n-1}\mathrm{Min}(\bar{P}_j(r),\bar{P}_{j+1}(r)) - \sum_{j=2}^{n-1}\bar{P}_j(r). \tag{4.28a}$$

Proof. Straightforward from the previous lemma. □

Equation (4.28a) is the key to the following observation:

PROPOSITION [4-5]. In the above setting, the futures price of the quality contract on the n zero coupon bonds is

$$\begin{aligned} Q(r,t;T,s_1,\ldots,s_n) &= \sum_{j=1}^{n-1} Q(r,t;T,s_j,s_{j+1}) - \sum_{j=2}^{n-1} k_j F(r,t;T,s_j) \\ &= k_1 F(r,t;T,s_1)\cdot N(d_{1,2}) \\ &\quad + \sum_{i=2}^{n-1} k_i\, F(r,t;T,s_i)\cdot [N(d_{i,i-1}) + N(d_{i,i+1}) - 1] \\ &\quad + k_n F(r,t;T,s_n)\cdot N(d_{n,n-1}), \end{aligned} \tag{4.29}$$

where $d_{i,j}$ (now defined for $i \neq j$ in $\{1,\ldots,n\}$) and $Q(r,t;T,s_j,s_{j+1})$ are as in the two-bond case.

Proof. Propositions [4-2] and [4-3] imply that the expression after the first equality is indeed a futures price. The second equality is a matter of algebra. The desired boundary condition is satisfied because of (4.28a). □

5. The Quality Futures Contract on n Commodities, with Vasicek Interest Rate Uncertainty

In this section, another Black–Scholes-type formula will be derived, in a setting where the quality option is on n commodities, and interest rates are stochastic. This setting is somewhat similar to the one in the option pricing (one-stock) model of Rabinovitch (1989). (The latter paper is not needed in order to understand the analysis below.) The assumptions are as follows:

(i) Bond prices and their derivative securities satisfy the assumptions of the Vasicek model as in the previous section. In what follows it would be typographically more convenient to write the time variable as an argument, and also to slightly change the notation in (4.1) to

$$\mathrm{d}r(t) = \alpha(r_0 - r(t))\,\mathrm{d}t + \sigma_0\,\mathrm{d}B_0(t), \tag{5.1}$$

where σ_0 is a constant, $\{B_0(t); t \geq 0\}$ is a standard Brownian motion. (The rest of the notation will be as before. In particular, the Brownian motions in (5.1) and later in (5.2) are relative to a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where $\mathcal{P}$ is interpreted as the objective probability measure.)

(ii) The positive price processes $\{S_i(t); t \geq 0\}$ $(i = 1,\ldots,n)$ of n given commodities evolve according to

$$\mathrm{d}S_i(t) = \nu_i(S_i(t), t)\,\mathrm{d}t + \sigma_i S_i(t)\,\mathrm{d}B_i(t), \tag{5.2}$$

where σ_i are constants, ν_i are suitably well-behaved functions and $\{B_i(t); t \geq 0\}$ are standard Brownian motions such that (informally) $\mathrm{d}B_i(t) \cdot \mathrm{d}B_j(t) = \rho_{i,j}\,\mathrm{d}t$ for some constants $\rho_{i,j} \in (-1, 1)$, $\rho_{i,i} = 1$ $(i \neq j \in \{0, 1, \ldots, n\})$. The commodities may be viewed as different grades of the same grain, but for simplicity we assume zero storage costs and no convenience yield. The following related notation will be used later, for $m = 1, \ldots, n$ and $i, j \in \{1, \ldots, m-1, m+1, \ldots, n\}$:

$$C^m_{i,j} = \sigma_i\sigma_j\rho_{i,j} - \sigma_j\sigma_m\rho_{j,m} - \sigma_i\sigma_m\rho_{i,m} + \sigma_m^2, \tag{5.3}$$

and C^m will denote the positive-definite $(n-1) \times (n-1)$-matrix with those elements.[5]

Regarding $X(t) \equiv (r(t), S_1(t), \ldots, S_n(t))$ as a state variable vector, the matrix η from (3.1) is diagonal with elements (counting rows and columns by $i = 0, 1, \ldots, n$) $\eta_{00} = \sigma_0$ and $\eta_{ii} = \sigma_i S_i$ $(i = 1, \ldots, n)$, and thus $\chi = \eta\rho\eta'$ has elements

$$\chi_{i,j} = \rho_{i,j}\eta_{i,i}\eta_{j,j} \quad (i, j = 0, 1, \ldots, n). \tag{5.4}$$

(See Remark 1 in Section 3). The $n+1$ elements of the vector μ in the futures PDE (3.2) are $\mu_0 = \alpha(\bar{r} - r)$ from Section 4 and $\mu_i = rS_i$ $(i = 1, \ldots, n)$, where the latter assertion follows from the fact that $\{S_i\}$ are prices of traded assets. (See Appendix C for a more formal argument.) This can be substituted in the futures PDE (3.2), where now $H = H(r, S_1, \ldots, S_n, t)$, and one obtains

$$H_t + \alpha(\bar{r} - r)H_r + r\sum_{i=1}^{n} S_i H_{S_i} + \frac{1}{2}\left\{\sigma_0^2 H_{rr} + \sigma_0\sum_{i=1}^{n}\sigma_i\rho_{0,i}S_i H_{rS_i} + \sum_{i,j=1}^{n}\sigma_i\sigma_j\rho_{i,j}S_iS_jH_{S_iS_j}\right\} = 0. \tag{5.5}$$

Let us start with the futures price on a single commodity.

PROPOSITION [5-1]. Consider the futures contract on commodity $i \in \{1, \ldots, n\}$ expiring at time T (i.e., with a boundary condition $S_i(T)$ at time T). Then its futures price at time $t < T$, given that $S_i(t) = S_i$ and $r(t) = r$, is

$$f_i(r, S_i, t; T) \equiv S_i A_0(t, T)\exp(-r_i^*(b(t, T) - (T - t)))\exp(rb(t, T)), \tag{5.6}$$

where A_0 and b are as in (4.7)–(4.8) (with $\sigma = \sigma_0$) and

$$r_i^* \equiv \bar{r} + \sigma_0\sigma_i\rho_{0,i}/\alpha = r_0 - \sigma_0(\lambda - \sigma_i\rho_{0,i})/\alpha. \tag{5.7}$$

Proof. Take for example $i = 1$. Let us try a solution of the form $H = S_1\phi(r,t)$ with $\phi(r,T) = 1$. Substituting this in the PDE (5.5) and then dividing by S_1 yields

$$\phi_t + \alpha(r_1^* - r)\phi_r + r\phi + \frac{1}{2}\sigma_0^2\phi_{rr} = 0, \quad \phi(r,T) = 1, \tag{5.8}$$

where r_1^* is as in (5.7). The solution of a somewhat more general problem is obtained in Appendix B. (In that appendix, take $\kappa = 1$, $\hat{r} = r_1^*$, $\sigma = \sigma_0$, and rearrange to obtain (5.6).) □

Recalling that the forward price is $S_i/P(r,t;T)$, it can be shown that Equation (5.6) is consistent with the forward-futures relationship (1.6) in Jamshidian (1993), which is obtained via the equivalent martingale measure technique.

Next, let us apply Proposition [3-1] with $J^i = f_i(r, S_i, t; T)$, $i = 1, \ldots, n$. Here we have, with the notation of Section 3, for $i, j = 1, \ldots, n-1$,

$$y^i \equiv f_i/f_n, \quad c_{i,j} = y^i y^j C_{i,j}^n, \tag{5.9}$$

where $C_{i,j}^n$ is as in (5.3). Indeed, by (5.6) one can write $y^i = S_i\gamma_{i,n}(t)/S_n$, where $\gamma_{i,n}$ is some function depending only on t. The row vector y_x^1 from Section 3 is now

$$(y_r^1, y_{S_1}^1, \ldots, y_{S_n}^1) = (0, y^1/S_1, 0, \ldots, 0, -y^1/S_n), \tag{5.10}$$

and similarly for the other y^i's. Substituting them as well as χ from (5.4) in Equation (3.6), it follows that the $(n-1)\times(n-1)$-matrix c from the transformed PDE is as in (5.9).

To solve for the quality futures price, it is instructive, as in the previous section, to start with the special case $n = 2$. Our closed-form formula below is analogous, after simple algebra, to the option pricing formula (11′) in Stulz (1982).

PROPOSITION [5-2]. For $t < T$, let $f_i(r, S_i, t; T)$ be the T-expiration futures price at time t, given that $S_i(t) = S_i$ and $r(t) = r$, on commodity $i \in \{1, 2\}$, and suppose $1/k_i$ is the corresponding conversion factor in a quality futures contract on these two commodities. Then the quality futures price is

$$\begin{aligned} q(r, S_1, S_2, t; T) = \\ k_1 f_1(r, S_1, t; T) \cdot N\left(\frac{\ln\dfrac{k_2 f_2(r, S_2, t; T)}{k_1 f_1(r, S_1, t; T)} - \dfrac{1}{2}\Lambda^2(t,T)}{\Lambda(t,T)}\right) \\ + k_2 f_2(r, S_2, t; T) \cdot N\left(\frac{\ln\dfrac{k_1 f_1(r, S_1, t; T)}{k_2 f_2(r, S_2, t; T)} - \dfrac{1}{2}\Lambda^2(t,T)}{\Lambda(t,T)}\right) \end{aligned} \tag{5.11}$$

where

$$\Lambda^2(t,T) \equiv (\sigma_1^2 - 2\rho_{1,2}\sigma_1\sigma_2 + \sigma_2^2)(T-t) \equiv \bar{\sigma}^2(T-t).$$

Proof. In light of the preceding discussion, specialized to $y \equiv y^1 \equiv f_1/f_2$, the PDE (3.9′) becomes

$$h_t(y,t) + \frac{1}{2}\bar{\sigma}^2 y^2 h_{yy}(y,t) = 0. \tag{5.12}$$

The argument is completed like in the proof of Proposition [4-3]. □

Let us now turn to the n-commodity case. First, we note that the technique from the previous section does not work here, as it relies on the fact that bond prices depend on a one-dimensional state variable r. We can still reduce the dimensionality of the problem by applying the technique from Section 3. We obtain:

PROPOSITION [5-3]. The futures price of the quality contract on n commodities (with notation as in the previous proposition) is

$$\begin{aligned} &q(r, S_1, \ldots, S_n, t; T) \\ &\quad = \sum_{m=1}^{n} k_m f_m(r, S_m, t; T) \cdot \Phi_{n-1}(d^m; C^m(T-t)), \end{aligned} \tag{5.13}$$

where the matrix C^m is as in (5.3), and d^m is the $(n-1)$-vector whose coordinates, indexed by $i \in \{1, \ldots, m-1, m+1, \ldots, n\}$, are

$$d_i^m = \ln[k_i f_i(r, S_i, t; T)/k_m f_m(r, S_m, t; T)]. \tag{5.14}$$

$\Phi_{n-1}(\cdot;\Gamma)$ (and $\phi_{n-1}(\cdot;\Gamma)$ below) represent the normal $(n-1)$-variate cumulative distribution function (and density function, respectively) with covariance matrix Γ and vector of means equal to $\frac{1}{2} \times$ (diagonal of Γ).

Proof. The solution of the PDE (5.5) is the integral of the fundamental solution (Green function) G against the boundary condition, which is

$$\bar{H}(r,S) \equiv \min(k_1 S_1, \ldots, k_n S_n) = \sum_{m=1}^{n} k_m \bar{H}_m(S), \tag{5.15}$$

where $\bar{H}_m(S) \equiv S_m \cdot 1_{L_m}(S)$ and $L_m \equiv \{s \in (0,\infty)^n;\, k_m s_m = \min(k_1 s_1, \ldots, k_n s_n)\}$. We use 1 to denote a characteristic function.[6] We claim that this decomposition of the boundary condition corresponds exactly to the n terms in (5.13), namely that for each m

$$\int_{\mathbb{R}\times(0,\infty)^n} G(r,S,t;r',S',T) \cdot \bar{H}_m(S')\,\mathrm{d}r'\,\mathrm{d}S'$$

$$= f_m(r, S_m, t; T) \cdot \Phi_{n-1}(d^m; C^m \tau), \tag{5.16}$$

where $\tau \equiv T - t$. Proving that will clearly establish our assertion. for reasons of symmetry, it is sufficient to show that for one m, say $m = n$. (We will sometimes omit the dependence on n.)

To that end, we will use Remark 4 in Section 3, now with the LHS from (5.16). Note that it requires, as a sufficient condition, that the boundary functions are continuous. However, we will accept, heuristically, that it also holds for appropriate characteristic functions. Let us use Proposition [3-1] with y^i from (5.9), i.e., with $J^n = f_n$ as a numeraire, to transform the PDE (5.5) into (3.9′), now with the particular coefficients from (5.9). Here it is convenient to change variables once again to $z = \ln(y)$ (coordinate-by-coordinate), and the PDE in $\xi(z,t) = h(y(z), t)$ becomes

$$\frac{\partial \xi}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n-1} C^n_{i,j} \frac{\partial^2 \xi}{\partial z_i \partial z_j} - \frac{1}{2} \sum_{j=1}^{n-1} C^n_{j,j} \frac{\partial \xi}{\partial z_j} = 0. \tag{5.17}$$

The well-known fundamental solution is $\phi_{n-1}(z - z'; C^n \tau)$.

We wish to calculate the integral from the RHS of (3.12). The above boundary value $\bar{H}_n$ is mapped under the transformation into a function denoted $\bar{\xi}$. More specifically, $\bar{H}_n(S) = S_n \bar{\xi}(z)$ for S and z which are related via $z_i = \ln(S_i/S_n)$. In our case $\bar{\xi} = 1_{\mathcal{H}}$, where $\mathcal{H} \equiv (K_1, \infty) \times \cdots \times (K_{n-1}, \infty)$ and $K_i = \ln(k_n/k_i)$. (Below, K will be the vector with those coordinates.) Thus, after the additional change of variables, the above integral becomes

$$\int \phi_{n-1}(z - z'; C^n \tau) 1_{\mathcal{H}}(z') \,\mathrm{d}z' = \Phi_{n-1}(-K + z; C^n \tau), \tag{5.18}$$

where the equality is straightforward. The argument is completed by applying (3.12). □

Our closed-form formula is analogous to the option pricing Equation (7) (with $X = 0$) in Johnson (1987), who uses risk-neutral valuation techniques. Again, this technique can be used as an alternative approach to obtain this section's results.

6. Discussion

The limitations of the Vasicek model in representing the term structure of interest rates are well known. However, our purpose in this paper is not to claim realism, but rather to provide simple tractable 'base case examples', in the spirit of Jamshidian (1989), which will be insightful in understanding the quality option in general. Among all such models in the literature with random interest rates and with marking-to-market, ours are clearly the simplest and most elegant. Several observations are worth pointing out:

1. In principle, all underlying assets have some probability of being cheapest-to-deliver at the expiration time, and this is reflected in the fact that all underlying straight futures prices appear in the formula ((4.29) or (5.13)). This is in contrast to many instances in the literature where the cost-of-carry model is used, relying only on one 'current cheapest-to-deliver' underlying asset.
2. Note the algebraic identity (with the notation of Section 5, just for concreteness):

$$k_i S_i(T) - \min_j \{k_j S_j(T)\} = \max_j \{[k_i S_i(T) - k_j S_j(T)^+\},$$
$$i = 1, \ldots, n. \tag{6.1}$$

   The RHS is naturally interpreted as an exchange-into-asset-i payoff (used by Hemler (1990)), with a prescribed adjustment in the number of units. This time-T relationship entails three time-t relationships, between spot, forward and futures prices, respectively. In particular, the time-t 'quality *forward* price', namely the forward price associated with $\text{Min}_j \{k_j S_j(T)\}$, must be equal to the forward price (i.e., to the spot price divided by $P(r, t; T)$) of the position 'long k_i units of asset i and short the exchange-into-asset-i payoff'. Several publications, e.g., Gay and Manaster (1984) and Hemler (1990), use this approach to analyze the quality futures contract, utilizing the well-known formula from Margrabe (1978) for the (two-asset) exchange option price. Strictly speaking, this is valid only for the quality forward price, which is a good approximation of the quality futures price when the interest rate volatility is small. In this paper the quality futures price is calculated exactly.
3. It is common practice in the literature to compute 'the value of the quality option', a concept which was not needed in this paper. In the setting from Section 5, here with abbreviated notation, the time-t futures price of the LHS of (6.1), namely $k_i f_i(t) - q(t)$, can naturally be regarded as a measure of the time-t 'impact of the quality option relative to the i-th straight futures contract'. Note that it is not a (spot) option price, but, rather, the *futures* price of the exchange-into-asset-i payoff from the RHS of (6.1). Ritchken and Sankarasubrahmanian (1992, p. 206) look at the quantity (in our notation) $\min_i k_i f_i(t) - q(t)$, which they term 'the value of the quality option'. Actually, this does not represent a price process (spot, forward or futures) of any time-T payoff. Perhaps a better name would be 'the impact of the quality option'. As for the 'quality option' as a stand-alone concept, it properly reflects the choice that the short has at delivery, but it is also potentially misleading, as the effect of this choice on the futures price process cannot be regarded as an option price.
4. Our closed-form formulas for the quality futures price have both the same appearance: It is a summation where the i-th term is equal to the i-th straight futures price times some 'coefficient'. As it is explained in Bick (1994), these coefficients represent a dynamic futures strategy in the underlying contracts which replicates the instant-by-instant cash flows of the futures contract

endowed with the quality option. The interpretation is thus analogous to the Black–Scholes formula, except that the traded assets have zero value and the replicated cash flow is continuous over time.

7. Summary

A futures contract with a quality option on n assets may be viewed as an ordinary cash-settlement contract with a certain boundary condition reflecting this option. This boundary condition, which is a function of the asset prices, may then be represented as a function of the n 'straight' futures prices on the single assets. The paper derives a PDE for the futures price of such a contract, where $n-1$ normalized futures prices serve as the state variables. This is used to obtain two closed-form formulas, in two different settings, for the futures price in the presence of the quality option.

Notes

[1] In practice this occurs at the end of every business day, relative to the settlement futures price (or relative to the futures price at which a contract was offset). However, in the continuous-time model in the next section, this will be done every instant.
[2] This is not the only way to adjust the invoice amount. For example, the wheat contract traded on the CBOT has a quality option with an additive price adjustment. See, e.g., Siegel and Siegel (1990), p. 418.
[3] In general, it is straightforward to verify that tr$[AB]$ = tr$[BA]$, if A is $m \times n$ and B is $n \times m$. For a reference, see Johnson and Schreiner (1996).
[4] See Equation (27) in Vasicek, or Jamshidian (1989), Equation (6). See also Remark 6 in Appendix B.
[5] One interpretation: If $X_1, \ldots, X_m$ are random variables with covariance matrix $(\sigma_i \sigma_j \rho_{i,j})$, then C^m is the covariance matrix of $X_i - X_m$, $i \in \{1, \ldots, m-1, m+1, \ldots, n\}$.
[6] Strictly speaking, (5.15) is valid at points S at which the minimum is obtained only for one m. Arguing heuristically, we can neglect the complement of this set because there is zero probability that the vector random variable $S(T)$ will assume values there.
[7] See footnote 3.

Appendix A: Some Mathematical Properties of the Futures Differential Operator

In the paper we regard futures prices merely as solutions of the futures PDE, and in this appendix we outline a few properties of the corresponding differential operator which are needed in Section 3. They are worth recording in their own right. Let

$$\mathcal{L}H \equiv H_t + H_x \mu + \frac{1}{2}\,\mathrm{tr}[\chi H_{xx}], \tag{A.1}$$

where here $\mu : \mathbb{R}^k \times I \to \mathbb{R}^k$ and $\chi : \mathbb{R}^k \times I \to \mathbb{R}^{k \times k}$ are allowed to be rather general, except that χ is a symmetric matrix. $\mathcal{L}$ is defined on elements of $C^{2,1}$, namely functions $H : \mathbb{R}^k \times [0, T) \to \mathbb{R}$ which are twice continuously differentiable in the 'state variables' $x \in \mathbb{R}^n$ and and once continuously differentiable in the 'time variable' $t \in [0,T)$. (The matrix and differentiation notations are as in Section 3.) In what follows we will simply refer to such a function, or to a vector function with $C^{2,1}$ coordinates, as 'suitably smooth'. For $H: \mathbb{R}^k \times [0,T] \to R$ this will mean that in addition $H(x,\cdot)$ is continuous for each x.

LEMMA [A-1]. Suppose $f, g: \mathbb{R}^k \times [0,T) \to R$ are both suitably smooth. Then

$$\mathcal{L}(fg) = f\mathcal{L}g + g\mathcal{L}f + \text{tr}[\chi f_x' g_x]. \tag{A.2}$$

Proof (outline): Substitute in $\mathcal{L}(fg)$ the (matrix) chain-rule identities: $(fg)_t = f_t g + f g_t$,

$$(fg)_x = gf_x + fg_x, \quad \text{and} \quad (fg)_{xx} = fg_{xx} + f_x' g_x + g_x' f_x + g f_{xx}.$$

Note that

$$\text{tr}[\chi g_x' f_x] = \text{tr}[g_x' f_x \chi] = \text{tr}[(g_x' f_x \chi)'] = \text{tr}[\chi f_x' g_x],$$

where the first equality follows from a general property of the trace[7] and the last equality follows from the symmetry of χ. □

LEMMA [A-2]. Suppose $J^i: \mathbb{R}^k \times [0,T) \to \mathbb{R}$, $i = 1,\ldots,n$, where $n \geq 2$, satisfy $\mathcal{L}J^i = 0$, and suppose $J^n > 0$. Define $y: \mathbb{R}^k \times [0,T) \to \mathbb{R}^{n-1}$ by

$$y^i(x,t) \equiv J^i(x,t)/J^n(x,t), \quad i = 1,\ldots,n-1, \tag{A.3}$$

so that (by (A.2))

$$0 = \mathcal{L}J^i = J^n \mathcal{L}y^i + \text{tr}\,[\chi (y_x^i)' J_x^n]. \tag{A.4}$$

As before, y_x^i is the row vector with elements $y_{x_j}^i = \partial y^i/\partial x_j$, $j = 1,\ldots,k$ (and similarly for J_x^n) and the matrix with rows y_x^i ($i = 1,\ldots,n-1$) will be denoted y_x.

Let $h: \mathbb{R}^{n-1} \times [0,T) \to \mathbb{R}$ be any suitably smooth function. In this case:

(a) $\psi(x,t) \equiv h(y(x,t),t)$ satisfies

$$\mathcal{L}\psi = h_t - \frac{1}{J^n}\text{tr}[\chi y_x' h_y' J_x^n] + \tfrac{1}{2}\text{tr}[y_x \chi y_x' h_{yy}]. \tag{A.5}$$

(b) $H(x,t) \equiv J^n(x,t) h(y(x,t),t)$ satisfies

$$\mathcal{L}H = J^n\{h_t + \tfrac{1}{2}\text{tr}[y_x \chi y_x' h_{yy}]\}. \tag{A.6}$$

In particular, $\mathcal{L}H = 0$ if and only if

$$h_t + \tfrac{1}{2}\text{tr}[y_x \chi y_x' h_{yy}] = 0. \tag{A.7}$$

Proof.

(a) To express $\mathcal{L}\psi$ in terms of h and y, substitute $\psi_t = h_t + h_y y_t$, $\psi_x = h_y y_x$, $\psi_{xx} = \Gamma + y_x' h_{yy} y_x$, where Γ is the $k \times k$-matrix whose (m,l)-element is $\sum_i h_{y_i} y^i_{x_m x_l}$. This gives:

$$\begin{aligned}\mathcal{L}\psi &= h_t + h_y y_t + h_y y_x \mu + \tfrac{1}{2}\,\mathrm{tr}[\chi\Gamma] + \tfrac{1}{2}\,\mathrm{tr}[\chi y_x' h_{yy} y_x] \\ &= h_t + \sum_i h_{y_i}(y^i_t + y^i_x \mu + \tfrac{1}{2}\,\mathrm{tr}[\chi y^i_{xx}]) + \tfrac{1}{2}\,\mathrm{tr}[\chi y_x' h_{yy} y_x] \\ &= h_t - \frac{1}{J^n}\sum_i h_{y_i}\,\mathrm{tr}[\chi (y^i_x)' J^n_x] + \tfrac{1}{2}\,\mathrm{tr}[\chi y_x' h_{yy} y_x],\end{aligned}$$

where in the second equality we substituted $\mathrm{tr}[\chi\Gamma] = \sum_i h_{y_i}\,\mathrm{tr}[\chi y^i_{xx}]$ (obtained by matrix algebra), and in the third equality (A.4) was used. Concluding the argument is again a matter of algebra.

(b) Applying Lemma [A-1] to $H = \psi J^n$ gives

$$\mathcal{L}H = 0 + J^n \mathcal{L}\psi + \mathrm{tr}[\chi \psi_x' J^n_x].$$

Substituting $\mathcal{L}\psi$ from part (a) and $\psi_x = h_y y_x$ gives the desired result. □

Appendix B: A Solution of a Vasicek-Type PDE

PROPOSITION [B-1]. For constants $\alpha, \kappa \in \mathbb{R}$ and continuous functions $\hat{r}(\cdot), \sigma(\cdot)$, consider the PDE in $U(r,t)$

$$U_t + \alpha(\hat{r}(t) - r)U_r + \kappa r U + \tfrac{1}{2}\sigma^2(t) U_{rr} = 0, \tag{B.1}$$

with the boundary condition $U(r,T) = 1$. Then, provided that $\alpha \neq 0$, the solution is

$$U(r,t) = \exp(\kappa b(t) r + \kappa \psi(t)), \tag{B.2}$$

where (omitting the dependence on T)

$$b(t) \equiv \alpha^{-1}[1 - \exp(\alpha(t-T))], \quad \psi(t) \equiv \int_t^T [\alpha \hat{r}(u) b(u) + \tfrac{1}{2}\kappa\sigma^2(u) b^2(u)]\,\mathrm{d}u. \tag{B.3}$$

If $\hat{r}$ and σ are constants, then, by straightforward calculation,

$$\psi(t) = \left(\hat{r} + \frac{\kappa\sigma^2}{2\alpha^2}\right)(T - t - b(t)) - \frac{\kappa\sigma^2}{4\alpha}\cdot b^2(t). \tag{B.4}$$

REMARK 6. The asset PDE (4.5) in the Vasicek setting is a special case, corresponding to $\kappa = -1$ and an appropriate choice of constants $\alpha, \hat{r}, \sigma$. The above

$U(r,t)$ from (B.2) is then identical to the bond price in (4.6). The case $\kappa = 1$ (with a different choice of $\hat{r}$ but the same α and σ) corresponds to the futures pricing PDE (5.8) from Section 5.

Outline of the proof. Trying $U = \exp(\varphi(t)r + \theta(t))$, the PDE gives

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t} - \alpha\varphi + \kappa\right) r + \frac{\mathrm{d}\theta}{\mathrm{d}t} + \alpha\hat{r}\varphi + \frac{1}{2}\sigma^2\varphi^2 = 0. \tag{B.5}$$

First, solving $\mathrm{d}\varphi/\mathrm{d}t - \alpha\varphi + \kappa = 0$ with $\varphi(T) = 0$, one obtains that φ is equal to $\kappa b(t)$. Then, integrating $\mathrm{d}\theta/\mathrm{d}t = -\alpha\hat{r} \cdot \kappa b(t) - \frac{1}{2}\sigma^2 \cdot \kappa^2 b^2(t)$ such that $\theta(T) = 0$ one obtains $\theta = \kappa\psi(t)$. □

Appendix C: Further Details Regarding the Futures PDE

TO SECTION 3

In the D-S setting, $R(X_t, t)$ is the instantaneous interest rate, and the vectors $\mathcal{S}(X_t, t)$ and $\Delta(X_t, t)$ represent the prices and dividend processes, respectively, of k given risky securities. It is assumed that R, $\mathcal{S}$ and Δ are suitably smooth on $\mathbb{R}^k \times I$ and that the partial derivative matrix $\mathcal{S}_x$ is non-singular. Regarding all these functions as known, Equation (1) in D-S gives the expression for μ which appears in the futures PDE (3.2) above. Interpreted coordinate-by-coordinate for each security,

$$\begin{aligned}&\mathcal{S}_x(x,t)\mu(x,t)\\ &\quad = R(x,t)\mathcal{S}(x,t) - \Delta(x,t) - \mathcal{S}_t(x,t) - \tfrac{1}{2}\,\mathrm{tr}[\chi(x,t)\mathcal{S}_{xx}(x,t)].\end{aligned} \tag{C.1}$$

D-S do not write the PDE (3.2) explicitly, but it follows easily from their Equation (23) for the futures price, which is in the form of a Feynman-Kac representation. For the related mathematics, see Appendix E in Duffie (1996).

One interpretation of μ is this: For each of the given k securities, consider the price of the continuously-reinvested (in itself) position, enumerated in units of the 'savings account'. Then, under an equivalent probability measure $\mathcal{Q}$ with respect to which these processes are martingales, μ is the drift of X, and it can be computed from the Girsanov kernel which transforms $\mathcal{P}$ into $\mathcal{Q}$. Again, see Duffie (1996) for the general approach. The PDE (3.2) is then the expression of the fact that $H(X_t, t)$ is a martingale under $\mathcal{Q}$, and hence, utilizing the Markovian structure and Itô's lemma, its drift in this case must be zero.

TO SECTION 5

In the futures PDE (5.5), the coefficients $(\mu_0, \mu_1, \ldots, \mu_n)'$ of $(H_r, H_{S_1}, \ldots, H_{S_n})$ were computed as follows: Let us take, just for simplicity, $n = 2$, and let

$\mathcal{S}(r, S_1, S_2) = (P, S_1, S_2)'$, where $P(r, t; T)$ is the price of the T-maturity zero coupon bond. Then (C.1) becomes

$$\begin{bmatrix} P_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \end{bmatrix} = r \begin{bmatrix} P \\ S_1 \\ S_2 \end{bmatrix} - \begin{bmatrix} P_t \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \sigma_0^2 P_{rr} \\ 0 \\ 0 \end{bmatrix}.$$

Thus $\mu_i = rS_i$, for $i = 1, 2$, and since P is a solution of (4.5), it follows that $\mu_0 = \alpha(\bar{r} - r)$.

Acknowledgements

A previous version of this paper was entitled ‘On the quality option and other homogeneous-of-degree 1 futures boundary conditions’, and was split into this paper and Bick (1994). The author acknowledges helpful comments by Dilip Madan, Haim Reisman and workshop participants at Simon Fraser University, The University of British Columbia, The Hebrew University of Jerusalem, Haifa University, the 1994 Cornell-Queen’s University derivative securities conference, the 1994 Northern Finance Association meetings and the January 1995 Econometric Society meetings. Valuable suggestions by an anonymous EFR referee and by the editor have helped improve the paper. The abbreviated term ‘the quality futures’ which is used in the paper was suggested by the referee.

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